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INFRARED STABILITY OF $\mathcal{N} = 2$ CHERN–SIMONS MATTER THEORIES

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ABSTRACT

According to the AdS4/CFT3 correspondence, $\mathcal{N} = 2$ supersymmetric Chern–Simons matter theories should have a stable fixed point in the infrared. In order to support this prediction we study RG flows of two–level Chern–Simons matter theories with/without flavors induced by the most general marginal superpotential compatible with $\mathcal{N} = 2$ supersymmetry. At two loops we determine the complete spectrum of fixed points and study their IR stability. Our analysis covers a large class of models including perturbations of the ABJM/ABJ theories with and without flavors, $\mathcal{N} = 2, 3$ theories with different CS levels corresponding to turning on a Romans mass and β –deformations. In all cases we find curves (or surfaces) of fixed points which are globally IR stable but locally unstable in the following sense: The system has only one direction of stability which in the ABJM case coincides with the maximal global symmetry preserving perturbation, whereas along any other direction it flows to a different fixed point on the surface. The question of conformal invariance vs. finiteness is also addressed: While in general vanishing beta–functions imply two–loop finiteness, we find a particular set of flavored theories where this is no longer true.

Keywords: Chern–Simons theories, $N = 2$ Supersymmetry, Infrared stability.

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1 Introduction

Recently, a renewed interest in three dimensional Chern–Simons (CS) theories has been triggered by the formulation of the $\text{AdS}_4/\text{CFT}_3$ correspondence between CS matter theories and M/string theory. While pure CS is a topological theory [1, 2], the addition of matter degrees of freedom makes it dynamical and can be used to describe nontrivial 3D systems. The addition of matter can be also exploited to formulate theories with extended supersymmetry [3, 4, 5]. Chern–Simons matter theories corresponding to a single gauge group can be at most $\mathcal{N} = 3$ supersymmetric [6], while the use of direct products of groups and matter in the bifundamental representation allows to increase supersymmetry up to $\mathcal{N} = 8$ [7].

This has led to the precise formulation of the $\text{AdS}_4/\text{CFT}_3$ correspondence which in its original form [8] states that M-theory on $\text{AdS}_4 \times S^7/\mathcal{Z}_k$ describes the strongly coupled dynamics of a two-level $\mathcal{N} = 6$ supersymmetric Chern–Simons theory with $U(N)_k \times U(N)_{-k}$ gauge group and $SU(2) \times SU(2)$ invariant matter in the bifundamental. This is the field theory generated at low energies by a stack of N M2-branes probing a $\mathcal{C}^4/\mathcal{Z}_k$ singularity. In the decoupling limit $N \rightarrow \infty$ with $\lambda \equiv N/k$ large and fixed, choosing $N \ll k^5$, the radius of the eleventh dimension in M-theory shrinks to zero and the dual description is given in terms of a type IIA string theory on $\text{AdS}_4 \times \mathcal{CP}^3$ background [8]. In the particular case of $N = 2$, supersymmetry gets enhanced to $\mathcal{N} = 8$ and the strings provide a dual description of the Bagger–Lambert–Gustavsson (BLG) model [7, 9, 10]. Enhancement of supersymmetry occurs also for $k = 1, 2$ where the ABJM theory describes the low energy dynamics of N membranes in flat space and in $\mathcal{R}^8/\mathcal{Z}_2$, respectively [8, 11, 12].

Since the original formulation of the correspondence, a lot of work has been done for studying the dynamical properties of this particular class of CS matter theories, such as integrability [13]–[23], the structure of the chiral ring and the operatorial content [8], [24]–[30] and dynamical supersymmetry breaking [31, 32]. Many efforts have been also devoted to the generalization of the correspondence to different gauge groups [34, 35], to less (super)symmetric backgrounds [33]–[43] and to include flavor degrees of freedom [44, 45, 46, 47, 48]. Theories with two different CS levels (k_1, k_2) have been also introduced [3, 42] which correspond to turning on a Romans mass [49] in the dual background.

CS matter theories involved in the $\text{AdS}_4/\text{CFT}_3$ correspondence are of course at their superconformal fixed point ⁴. Compactification of type IIA supergravity on $\text{AdS}_4 \times \mathcal{CP}^3$ does not contain scalar tachyons [50]. Since these states are dual to relevant operators in the corresponding field theory, $\text{AdS}_4/\text{CFT}_3$ correspondence leads to the prediction that in the far IR fixed points should be stable.

As a nontrivial check of the correspondence, it is then interesting to investigate the properties of these fixed points in the quantum field theory in order to establish whether they are isolated fixed points or they belong to a continuum surface of fixed points, whether they are IR stable and which are the RG trajectories which intersect them. Since

⁴A classification of a huge landscape of superconformal Chern–Simons matter theories in terms of matter representations of global symmetries has been given in [51].

for $k \gg N$ the CS theory is weakly coupled, a perturbative approach is available.

With these motivations in mind, we consider a $\mathcal{N} = 2$ supersymmetric two-level CS theory for gauge group $U(N) \times U(M)$ with matter in the bifundamental representation and flavor degrees of freedom in the fundamental, perturbed by the most general matter superpotential compatible with $\mathcal{N} = 2$ supersymmetry. For particular values of the couplings the model reduces to the $\mathcal{N} = 6$ ABJ/ABJM superconformal theories [8, 34] ($\mathcal{N} = 8$ BLG theory [7] for $N = M = 2$) or to the superconformal $\mathcal{N} = 2, 3$ theories with different CS levels studied in [42], in all cases with and without flavors. More generally, it describes marginal (but not exactly marginal) perturbations which can drive the theory away from the superconformal points.

At two loops, we compute the beta-functions and determine the spectrum of fixed points. In the absence of flavors the condition of vanishing beta-functions necessarily implies the vanishing of anomalous dimensions for all the elementary fields of the theory. Therefore, the set of superconformal fixed points coincides with the set of superconformal finite theories. When flavors are present this is no longer true and in the space of the couplings we determine a surface of fixed points where the theory is superconformal but not two-loop finite.

When flavors are turned off we determine a continuum surface of fixed points which contains as non-isolated fixed points the BLG, the ABJ and ABJM theories. The case of theories with equal CS levels and $U(1)_A \times U(1)_B$ symmetry preserving perturbations has been already investigated in [52]. The present paper provides details for that class of theories and generalizes the results to the case of no-symmetry preserving perturbations. When the CS levels are different the surface contains a $\mathcal{N} = 2$, $SU(2)_A \times SU(2)_B$ invariant and a $\mathcal{N} = 3$ superconformal theories. This result confirms the existence of the superconformal points conjectured in [42]. Moreover, we prove that the two theories are connected by a line of $\mathcal{N} = 2$ fixed points, as conjectured there.

We extend our analysis to the case of complex couplings, so including fixed points corresponding to beta-deformed theories [40].

In the presence of flavor matter the spectrum of fixed points spans a seven dimensional hypersurface in the space of the couplings which contains the fixed point corresponding to the ABJM/ABJ models with flavors studied in [44, 45, 46]. More generally, we find a fixed point which describes a $\mathcal{N} = 3$ theory with different CS levels with the addition of flavor degrees of freedom [45]. As a generalization of the pattern arising in the unflavored case, we find that it is connected by a four dimensional hypersurface of $\mathcal{N} = 2$ fixed points to a line of $\mathcal{N} = 2$ fixed points with $SU(2)_A \times SU(2)_B$ invariance in the bifundamental sector.

We then study RG trajectories around these fixed points in order to investigate their IR stability. The pattern which arises is common to all these theories, flavors included or not, and can be summarized as follows.

- Infrared stable fixed points always exist and we determine the RG trajectories which connect them to the UV stable fixed point (free theory).
- In general these fixed points belong to a continuum surface. The surface is globally

stable since RG flows always point towards it.

- Locally, each single fixed point has only one direction of stability which corresponds to perturbations along the RG trajectory which intersects the surface at that point. In the ABJ/ABJM case this direction coincides with the maximal flavor symmetry preserving perturbation [52]. Along any other direction, perturbations drive the system away from the original point towards a different point on the surface. This is what we call local instability.
- When flavors are added, stability is guaranteed by the presence of nontrivial interactions between flavors and bifundamental matter. The fixed point corresponding to setting these couplings to zero is in fact unstable.

The organization of the paper is the following: In Section 2 we introduce our general theory, we discuss its properties and quantize it in a manifest $\mathcal{N} = 2$ set-up. In Section 3 we compute the two-loop divergences, we renormalize the theory and determine the beta-functions. Sections 4 and 5 contain the main results of the paper concerning the determination of the spectrum of fixed points and the study of the IR stability for the most interesting cases. A concluding discussion follows. In the Appendix we list our conventions for 3D $\mathcal{N} = 2$ superspace.

2 $\mathcal{N} = 2$ Chern–Simons matter theories

In three dimensions, we consider a $\mathcal{N} = 2$ supersymmetric $U(N) \times U(M)$ Chern–Simons theory for vector multiplets (V, \hat{V}) coupled to chiral multiplets A^i and B_i , $i = 1, 2$, in the (N, \bar{M}) and (\bar{N}, M) representations of the gauge group respectively, and flavor matter described by two couples of chiral superfields Q_i, \tilde{Q}_i , $i = 1, 2$ charged under the gauge groups and under a global $U(N_f)_1 \times U(N'_f)_2$.

The vector multiplets V, \hat{V} are in the adjoint representation of the gauge groups $U(N)$ and $U(M)$ respectively, and we write $V_a^b \equiv V^A(T_A)^b_a$ and $\hat{V}_{\hat{a}}^{\hat{b}} \equiv \hat{V}^{\hat{A}}(T_{\hat{A}})^{\hat{b}}_{\hat{a}}$. Bifundamental matter carries global $SU(2)_A \times SU(2)_B$ indices $A^i, \bar{A}_i, B_i, \bar{B}^i$ and local $U(N) \times U(M)$ indices $A_{\hat{a}}^a, \bar{A}_{\hat{a}}^a, B_{\hat{a}}^a, \bar{B}_{\hat{a}}^a$. Flavor matter carries (anti)fundamental gauge and global indices, $(Q_1^r)^a, (\tilde{Q}_{1,r})_a, (Q_2^{r'})^{\hat{a}}, (\tilde{Q}_{2,r'})_{\hat{a}}$, with $r = 1, \dots, N_f$, $r' = 1, \dots, N'_f$.

In $\mathcal{N} = 2$ superspace the action reads (for superspace conventions see Appendix)

$$\mathcal{S} = \mathcal{S}_{\text{CS}} + \mathcal{S}_{\text{mat}} + \mathcal{S}_{\text{pot}} \quad (2.1)$$

with

$$\begin{aligned}\mathcal{S}_{\text{CS}} &= K_1 \int d^3x d^4\theta \int_0^1 dt \text{Tr} \left[V \bar{D}^\alpha (e^{-tV} D_\alpha e^{tV}) \right] \\ &+ K_2 \int d^3x d^4\theta \int_0^1 dt \text{Tr} \left[\hat{V} \bar{D}^\alpha (e^{-t\hat{V}} D_\alpha e^{t\hat{V}}) \right]\end{aligned}\quad (2.2)$$

$$\begin{aligned}\mathcal{S}_{\text{mat}} &= \int d^3x d^4\theta \text{Tr} \left(\bar{A}_i e^V A^i e^{-\hat{V}} + \bar{B}^i e^{\hat{V}} B_i e^{-V} \right) \\ &+ \int d^3x d^4\theta \text{Tr} \left(\bar{Q}_r^1 e^V Q_1^r + \bar{\tilde{Q}}^{1,r} \tilde{Q}_{1,r} e^{-V} + \bar{Q}_{r'}^2 e^{\hat{V}} Q_2^{r'} + \bar{\tilde{Q}}^{2,r'} \tilde{Q}_{2,r'} e^{-\hat{V}} \right)\end{aligned}\quad (2.3)$$

$$\begin{aligned}\mathcal{S}_{\text{pot}} &= \int d^3x d^2\theta \text{Tr} \left[h_1 (A^1 B_1)^2 + h_2 (A^2 B_2)^2 + h_3 (A^1 B_1 A^2 B_2) + h_4 (A^2 B_1 A^1 B_2) \right. \\ &\quad + \lambda_1 (Q_1 \tilde{Q}_1)^2 + \lambda_2 (Q_2 \tilde{Q}_2)^2 + \lambda_3 Q_1 \tilde{Q}_1 Q_2 \tilde{Q}_2 \\ &\quad \left. + \alpha_1 \tilde{Q}_1 A^1 B_1 Q_1 + \alpha_2 \tilde{Q}_1 A^2 B_2 Q_1 + \alpha_3 \tilde{Q}_2 B_1 A^1 Q_2 + \alpha_4 \tilde{Q}_2 B_2 A^2 Q_2 \right] + h.c.\end{aligned}\quad (2.4)$$

Here $2\pi K_1, 2\pi K_2$ are two independent integers, as required by gauge invariance of the effective action. In the perturbative regime we take $K_1, K_2 \gg N, M$. The superpotential (2.4) is the most general classically marginal perturbation which respects $\mathcal{N} = 2$ supersymmetry but allows only for a $U(N_f) \times U(N'_f)$ global symmetry in addition to a global $U(1)$ under which the bifundamentals have for example charges $(1, 0, -1, 0)$.

For generic values of the couplings, the action (2.1) is invariant under the following gauge transformations

$$e^V \rightarrow e^{i\bar{\Lambda}_1} e^V e^{-i\Lambda_1} \quad e^{\hat{V}} \rightarrow e^{i\bar{\Lambda}_2} e^{\hat{V}} e^{-i\Lambda_2} \quad (2.5)$$

$$\begin{aligned}A^i &\rightarrow e^{i\Lambda_1} A^i e^{-i\Lambda_2} & B_i &\rightarrow e^{i\Lambda_2} B_i e^{-i\Lambda_1} \\ Q_1 &\rightarrow e^{i\Lambda_1} Q_1 & \tilde{Q}_1 &\rightarrow \tilde{Q}_1 e^{-i\Lambda_1} \\ Q_2 &\rightarrow e^{i\Lambda_2} Q_2 & \tilde{Q}_2 &\rightarrow \tilde{Q}_2 e^{-i\Lambda_2}\end{aligned}\quad (2.6)$$

where Λ_1, Λ_2 are two chiral superfields parametrizing $U(N)$ and $U(M)$ gauge transformations, respectively. Antichiral superfields transform according to the conjugate of (2.6).

For special values of the couplings we can have enhancement of global symmetries and/or R-symmetry with consequent enhancement of supersymmetry. We list the most important cases we will be interested in.

Theories without flavors

Turning off flavor matter ($N_f = N'_f = 0, \alpha_j = \lambda_j = 0$) and setting

$$K_1 = -K_2 \equiv K \quad , \quad h_1 = h_2 = 0 \quad (2.7)$$

we have $\mathcal{N} = 2$ ABJM/ABJ-like theories already studied in [52]. In this case the theory is invariant under two global $U(1)$'s in addition to $U(1)_R$. The transformations are

$$\begin{aligned} U(1)_A : \quad A^1 &\rightarrow e^{i\alpha} A^1 & , & \quad U(1)_B : \quad B_1 \rightarrow e^{i\beta} B_1 \\ A^2 &\rightarrow e^{-i\alpha} A^2 & , & \quad B_2 \rightarrow e^{-i\beta} B_2 \end{aligned} \quad (2.8)$$

When $h_3 = -h_4 \equiv h$, the global symmetry becomes $U(1)_R \times SU(2)_A \times SU(2)_B$ and gets enhanced to $SU(4)_R$ for $h = 1/K$ [8, 33]. For this particular values of the couplings we recover the $\mathcal{N} = 6$ superconformal ABJ theory [34] and for $N = M$ the ABJM theory [8].

More generally, we can select theories corresponding to complex couplings

$$h_3 = h e^{i\pi\beta} \quad , \quad h_4 = -h e^{-i\pi\beta} \quad (2.9)$$

These are $\mathcal{N} = 2$ β -deformations of the ABJ-like theories. For particular values of h and β we find a superconformal invariant theory.

Going back to real couplings, we now consider the more general case $K_1 \neq -K_2$. Setting

$$h_1 = h_2 = \frac{1}{2} (h_3 + h_4) \quad (2.10)$$

the corresponding superpotential reads

$$\mathcal{S}_{\text{pot}} = \frac{1}{2} \int d^3x d^2\theta \text{Tr} [h_3 (A^i B_i)^2 + h_4 (B_i A^i)^2] + h.c. \quad (2.11)$$

This is the class of $\mathcal{N} = 2$ theories studied in [42] with $SU(2)$ invariant superpotential, where $SU(2)$ rotates simultaneously A^i and B_i .

When $h_3 = -h_4$, that is $h_1 = h_2 = 0$, we have the particular set of $\mathcal{N} = 2$ theories with global $SU(2)_A \times SU(2)_B$ symmetry [42]. This is the generalization of ABJ/ABJM-like theories to $K_1 \neq -K_2$. According to AdS/CFT, for particular values of $h_3 = -h_4$ we should find a superconformal invariant theory.

Another interesting fixed point should correspond to $h_3 = \frac{1}{K_1}$ and $h_4 = \frac{1}{K_2}$. The $U(1)_R$ R-symmetry is enhanced to $SU(2)_R$ and the theory is $\mathcal{N} = 3$ superconformal [42].

Theories with flavors

Setting

$$\begin{aligned} K_1 = -K_2 &\equiv K & , & \quad h_1 = h_2 = 0 & , & \quad h_3 = -h_4 = \frac{1}{K} \\ \lambda_1 &= \frac{a_1^2}{2K} & , & \quad \lambda_2 = -\frac{a_2^2}{2K} & , & \quad \lambda_3 = 0 \\ \alpha_1 = \alpha_2 &= \frac{a_1}{K} & , & \quad \alpha_3 = \alpha_4 = \frac{a_2}{K} \end{aligned} \quad (2.12)$$

with a_1, a_2 arbitrary, our model reduces to the class of $\mathcal{N} = 2$ theories studied in [44]. Choosing in particular $a_1 = -a_2 = 1$ there is an enhancement of R-symmetry and the

theory exhibits $\mathcal{N} = 3$ supersymmetry. This set of couplings should correspond to a superconformal fixed point [44, 45, 46].

In the more general case of $K_1 \neq -K_2$, in analogy with the unflavored case we consider the class of theories with

$$h_1 = h_2 = \frac{1}{2}(h_3 + h_4) \quad ; \quad \alpha_1 = \alpha_2 \quad , \quad \alpha_3 = \alpha_4 \quad (2.13)$$

For generic couplings these are $\mathcal{N} = 2$ theories with a $SU(2)$ symmetry in the bifundamental sector which rotates simultaneously A^i and B_i . When $h_3 = -h_4$ this symmetry is enhanced to $SU(2)_A \times SU(2)_B$. The flavor sector has only $U(N_f) \times U(N'_f)$ flavor symmetry.

Within this class of theories we can select the one corresponding to

$$\begin{aligned} \lambda_1 = \frac{h_3}{2} \quad , \quad \lambda_2 = \frac{h_4}{2} \quad , \quad \lambda_3 = 0 \\ \alpha_1 = \alpha_2 = h_3 \quad , \quad \alpha_3 = \alpha_4 = h_4 \end{aligned} \quad (2.14)$$

The values $h_3 = \frac{1}{K_1}, h_4 = \frac{1}{K_2}$ give the $\mathcal{N} = 3$ superconformal theory with flavors mentioned in [45]. It corresponds to flavoring the $\mathcal{N} = 3$ theory of [42].

We now proceed to the quantization of the theory in a manifest $\mathcal{N} = 2$ setup. In each gauge sector we choose gauge-fixing functions $\bar{F} = D^2 V$, $F = \bar{D}^2 V$ and insert into the functional integral the factor

$$\int \mathcal{D}f \mathcal{D}\bar{f} \Delta(V) \Delta^{-1}(V) \exp \left\{ -\frac{K}{2\alpha} \int d^3x d^2\theta \text{Tr}(ff) - \frac{K}{2\alpha} \int d^3x d^2\bar{\theta} \text{Tr}(\bar{f}\bar{f}) \right\} \quad (2.15)$$

where $\Delta(V) = \int d\Lambda d\bar{\Lambda} \delta(F(V, \Lambda, \bar{\Lambda}) - f) \delta(\bar{F}(V, \Lambda, \bar{\Lambda}) - \bar{f})$ and the weighting function has been chosen in order to have a dimensionless gauge parameter α . We note that the choice of the weighting function is slightly different from the four dimensional case [53] where we usually use $\int \mathcal{D}f \mathcal{D}\bar{f} \exp \left\{ -\frac{1}{g^2\alpha} \int d^4x d^4\theta \text{Tr}(ff) \right\}$.

The quadratic part of the gauge-fixed action reads

$$\begin{aligned} S_{CS} + S_{gf} \rightarrow \frac{1}{2} K_1 \int d^3x d^4\theta \text{Tr} V \left(\bar{D}^\alpha D_\alpha + \frac{1}{\alpha} D^2 + \frac{1}{\alpha} \bar{D}^2 \right) V \\ + \frac{1}{2} K_2 \int d^3x d^4\theta \text{Tr} \hat{V} \left(\bar{D}^\alpha D_\alpha + \frac{1}{\alpha} D^2 + \frac{1}{\alpha} \bar{D}^2 \right) \hat{V} \end{aligned} \quad (2.16)$$

and leads to the gauge propagators

$$\langle V^A(1) V^B(2) \rangle = -\frac{1}{K_1} \frac{1}{\square} \left(\bar{D}^\alpha D_\alpha + \alpha D^2 + \alpha \bar{D}^2 \right) \delta^4(\theta_1 - \theta_2) \delta^{AB} \quad (2.17)$$

$$\langle \hat{V}^A(1) \hat{V}^B(2) \rangle = -\frac{1}{K_2} \frac{1}{\square} \left(\bar{D}^\alpha D_\alpha + \alpha D^2 + \alpha \bar{D}^2 \right) \delta^4(\theta_1 - \theta_2) \delta^{AB} \quad (2.18)$$

In our calculations we will use the analog of the Landau gauge, $\alpha = 0$.

Expanding $S_{CS} + S_{gf}$ at higher orders in V, \hat{V} we obtain the interaction vertices. For two-loop calculations we need

$$\begin{aligned}
S_{CS} + S_{gf} \rightarrow & \frac{i}{6} K_1 f^{ABC} \int d^3 x d^4 \theta \left(\bar{D}^\alpha V^A V^B D_\alpha V^C \right) \\
& - \frac{1}{24} K_1 f^{ABE} f^{ECD} \int d^3 x d^4 \theta \left(\bar{D}^\alpha V^A V^B D_\alpha V^C V^D \right) \\
& + \frac{i}{6} K_2 f^{ABC} \int d^3 x d^4 \theta \left(\bar{D}^\alpha \hat{V}^A \hat{V}^B D_\alpha \hat{V}^C \right) \\
& - \frac{1}{24} K_2 f^{ABE} f^{ECD} \int d^3 x d^4 \theta \left(\bar{D}^\alpha \hat{V}^A \hat{V}^B D_\alpha \hat{V}^C \hat{V}^D \right) \quad (2.19)
\end{aligned}$$

The ghost action is the same as the one of the four dimensional $\mathcal{N} = 1$ case [53]

$$S_{gh} = \text{Tr} \int d^4 x d^4 \theta \left[\bar{c}' c - c' \bar{c} + \frac{1}{2} (c' + \bar{c}') [V, (c + \bar{c})] \right] + \mathcal{O}(V^2) \quad (2.20)$$

and gives ghost propagators

$$\langle \bar{c}'(1) c(2) \rangle = \langle c'(1) \bar{c}(2) \rangle = -\frac{1}{\square} \delta^4(\theta_1 - \theta_2) \quad (2.21)$$

and cubic interaction vertices

$$\frac{i}{2} f^{ABC} \int d^4 x d^4 \theta \left(c'^A V^B c^C + \bar{c}'^A V^B c^C + c'^A V^B \bar{c}^C + \bar{c}'^A V^B \bar{c}^C \right) \quad (2.22)$$

We now quantize the matter sector. From the quadratic part of the action (2.3) we read the propagators

$$\begin{aligned}
\langle \bar{A}_{\hat{a}}^a(1) A_{\hat{b}}^b(2) \rangle &= -\frac{1}{\square} \delta^4(\theta_1 - \theta_2) \delta_{\hat{b}}^{\hat{a}} \delta_a^b \\
\langle \bar{B}_{\hat{a}}^a(1) B_{\hat{b}}^b(2) \rangle &= -\frac{1}{\square} \delta^4(\theta_1 - \theta_2) \delta_{\hat{b}}^{\hat{a}} \delta_a^b \\
\langle (\bar{Q}_r^1)_a(1) (Q_1^q)^b(2) \rangle &= -\frac{1}{\square} \delta^4(\theta_1 - \theta_2) \delta_a^b \delta_r^q \\
\langle (\bar{\tilde{Q}}_{1,r})_a(1) (\bar{\tilde{Q}}^{1,q})^b(2) \rangle &= -\frac{1}{\square} \delta^4(\theta_1 - \theta_2) \delta_a^b \delta_r^q \quad r, q = 1, \dots, N_f \\
\langle (\bar{Q}_{r'}^2)_{\hat{a}}(1) (Q_2^{q'})^{\hat{b}}(2) \rangle &= -\frac{1}{\square} \delta^4(\theta_1 - \theta_2) \delta_{\hat{a}}^{\hat{b}} \delta_{r'}^{q'} \\
\langle (\bar{\tilde{Q}}_{2,r'})_{\hat{a}}(1) (\bar{\tilde{Q}}^{2,q'})^{\hat{b}}(2) \rangle &= -\frac{1}{\square} \delta^4(\theta_1 - \theta_2) \delta_{\hat{a}}^{\hat{b}} \delta_{r'}^{q'} \quad r', q' = 1, \dots, N'_f
\end{aligned} \quad (2.23)$$

From the expansion of (2.3) mixed gauge/matter vertices entering two-loop calculations



Figure 1: One-loop diagrams for scalar propagators.

are

$$\begin{aligned}
\mathcal{S}_{\text{mat}} \rightarrow & \int d^3x d^4\theta \, \text{Tr} \left(\bar{A} V A - \bar{A} A \hat{V} + \bar{B} \hat{V} B - \bar{B} B V \right) \\
& + \int d^3x d^4\theta \, \text{Tr} \left(\frac{1}{2} \bar{A} V V A + \frac{1}{2} \bar{A} A \hat{V} \hat{V} - \bar{A} V A \hat{V} + \frac{1}{2} \bar{B} \hat{V} \hat{V} B + \frac{1}{2} \bar{B} B V V - \bar{B} \hat{V} B V \right) \\
& + \int d^3x d^4\theta \, \text{Tr} \left(\bar{Q}_r^1 V Q_1^r - \bar{Q}^{1,r} \tilde{Q}_{1,r} V + \bar{Q}_{r'}^2 \hat{V} Q_2^{r'} - \bar{Q}^{2,r'} \tilde{Q}_{2,r'} \hat{V} \right) \\
& + \int d^3x d^4\theta \, \text{Tr} \left(\frac{1}{2} \bar{Q}_r^1 V V Q_1^r + \frac{1}{2} \bar{Q}^{1,r} \tilde{Q}_{1,r} V V + \frac{1}{2} \bar{Q}_{r'}^2 \hat{V} \hat{V} Q_2^{r'} + \frac{1}{2} \bar{Q}^{2,r'} \tilde{Q}_{2,r'} \hat{V} \hat{V} \right)
\end{aligned} \tag{2.24}$$

Pure matter vertices can be read from the superpotential (2.4).

3 Two-loop renormalization and β -functions

It is well known that even in the presence of matter chiral superfields the CS actions cannot receive loop divergent corrections [54, 55]. In fact, gauge invariance requires $2\pi K_1, 2\pi K_2$ to be integers, so preventing any renormalization except for a finite shift. In particular, for the $\mathcal{N} = 2$ case it has been proved [55] that even finite renormalization is absent.

Divergent contributions are then expected only in the matter sector. Since a non-renormalization theorem still holds for the superpotential (in $\mathcal{N} = 2$ superspace perturbative calculations one can never produce local, chiral divergent contributions) divergences arise only in the Kahler sector and lead to field functions renormalization.

In odd spacetime dimensions there are no UV divergences at odd loops. Therefore, the first non trivial tests for the perturbative quantum properties of the theory arise at two loops.

3.1 One loop results

We first compute the finite quantum corrections to the scalar and gauge propagators which then enter two-loop computations.

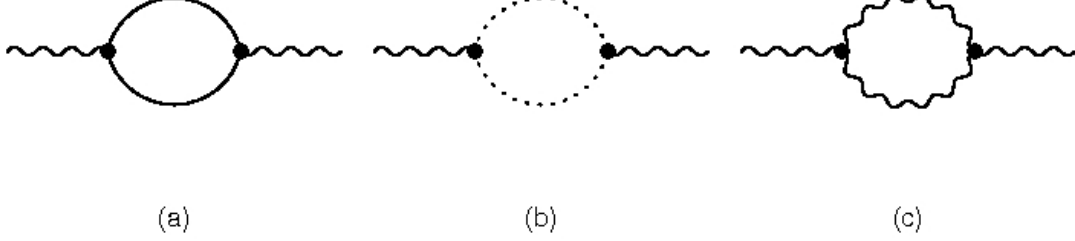


Figure 2: One-loop diagrams for gauge propagators.

The only diagrams contributing to the matter field propagators are the ones given in Fig. 1. It is easy to verify that they vanish for symmetry reasons.

We then move to the gauge propagator. Gauge one-loop self-energy contributions come from diagrams in Fig. 2 where chiral, gauge and ghost loops are present.

Performing the calculation in momentum space and using the superspace projectors [53]

$$\Pi_0 \equiv -\frac{1}{k^2} \{D^2, \bar{D}^2\}(k) \quad , \quad \Pi_{1/2} \equiv \frac{1}{k^2} \bar{D}^\alpha D^2 \bar{D}_\alpha(k) \quad \Pi_0 + \Pi_{1/2} = 1 \quad (3.1)$$

we find the following finite contributions to the quadratic action for the gauge fields

$$\begin{aligned} \Pi_{gauge}^{(1)}(2a) &= \frac{1}{8} f^{ABC} f^{A'BC} \int \frac{d^3 k}{(2\pi)^3} d^4 \theta \, B_0(k) k^2 V^A(k) \Pi_0 V^{A'}(-k) \\ \Pi_{gauge}^{(1)}(2b) &= -\frac{1}{8} f^{ABC} f^{A'BC} \int \frac{d^3 k}{(2\pi)^3} d^4 \theta \, B_0(k) k^2 V^A(k) (\Pi_0 + \Pi_{1/2}) V^{A'}(-k) \\ \Pi_{gauge}^{(1)}(2c) &= \left(M + \frac{N_f}{2} \right) \delta^{AA'} \int \frac{d^3 k}{(2\pi)^3} d^4 \theta \, B_0(k) k^2 V^A(k) \Pi_{1/2} V^{A'}(-k) \end{aligned} \quad (3.2)$$

$$\begin{aligned} \hat{\Pi}_{gauge}^{(1)}(2a) &= \frac{1}{8} \hat{f}^{ABC} \hat{f}^{A'BC} \int \frac{d^3 p}{(2\pi)^3} d^4 \theta \, B_0(p) p^2 \hat{V}^A(p) \Pi_0 \hat{V}^{A'}(-p) \\ \hat{\Pi}_{gauge}^{(1)}(2b) &= -\frac{1}{8} \hat{f}^{ABC} \hat{f}^{A'BC} \int \frac{d^3 p}{(2\pi)^3} d^4 \theta \, B_0(p) p^2 \hat{V}^A(p) (\Pi_0 + \Pi_{1/2}) \hat{V}^{A'}(-p) \\ \hat{\Pi}_{gauge}^{(1)}(2c) &= \left(N + \frac{N'_f}{2} \right) \delta^{AA'} \int \frac{d^3 p}{(2\pi)^3} d^4 \theta \, B_0(p) p^2 \hat{V}^A(p) \Pi_{1/2} \hat{V}^{A'}(-p) \end{aligned} \quad (3.3)$$

$$\tilde{\Pi}_{gauge}^{(1)}(2c) = -2\sqrt{NM} \delta^{A0} \delta^{A'0} \int \frac{d^3 p}{(2\pi)^3} d^4 \theta \, B_0(p) p^2 V^A(p) \Pi_{1/2} \hat{V}^{A'}(-p) \quad (3.4)$$

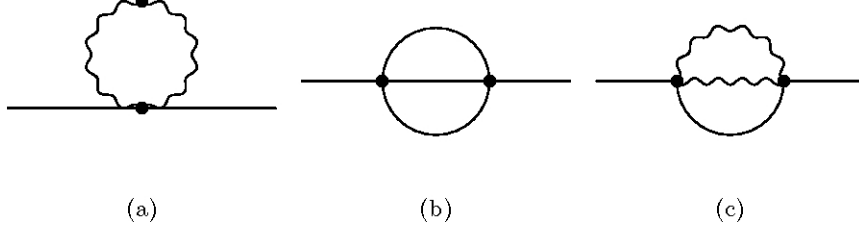


Figure 3: Two-loop divergent diagrams contributing to the matter propagators.

where $B_0(p) = 1/(8|p|)$ is the three dimensional bubble scalar integral (see (A.12)).

Summing all the contributions we see that the gauge loop cancels against part of the ghost loop as in the 4D $\mathcal{N} = 1$ case [56] and we find the known results [56, 57]

$$\begin{aligned}
\Pi_{gauge}^{(1)} &= \left[-\frac{1}{8} f^{ABC} f^{A'BC} + \left(M + \frac{N_f}{2} \right) \delta^{AA'} \right] \int \frac{d^3 p}{(2\pi)^3} d^4 \theta \, B_0(p) p^2 V^A(p) \Pi_{1/2} V^{A'}(-p) \\
\hat{\Pi}_{gauge}^{(1)} &= \left[-\frac{1}{8} f^{ABC} f^{A'BC} + \left(N + \frac{N'_f}{2} \right) \delta^{AA'} \right] \int \frac{d^3 p}{(2\pi)^3} d^4 \theta \, B_0(p) p^2 \hat{V}^A(p) \Pi_{1/2} \hat{V}^{A'}(-p)
\end{aligned} \tag{3.5}$$

together with $\tilde{\Pi}_{gauge}^{(1)}$ in (3.4) which mixes the two $U(1)$ gauge sectors.

3.2 Two-loop results

We are now ready to evaluate the matter self-energy contributions at two loops. Both for the bifundamental and the flavor matter the divergent diagrams are given in Fig. 3.

Evaluation of each diagram proceeds in the standard way by first performing D-algebra in order to reduce supergraphs to ordinary Feynman graphs and evaluate them in momentum space and dimensional regularization ($d = 3 - 2\epsilon$). Separating the contributions of each diagram, the results for the bifundamental matter are

$$\begin{aligned}
\Pi_{bif}^{(2)}(3a) &= - \left[\frac{1}{K_1^2} \left(2NM + NN_f - \frac{1}{2} (N^2 - 1) \right) + \right. \\
&\quad \left. + \frac{1}{K_2^2} \left(2NM + MN'_f - \frac{1}{2} (M^2 - 1) \right) + \frac{4}{K_1 K_2} \right] F(0) \, \text{Tr} (\bar{A}_i A^i + \bar{B}^i B_i) \\
\Pi_{bif}^{(2)}(3b) &= \left[4|h_1|^2 (MN + 1) + (|h_3|^2 + |h_4|^2) MN + (h_3 \bar{h}_4 + h_4 \bar{h}_3) \right. \\
&\quad \left. + (|\alpha_1|^2 NN_f + |\alpha_3|^2 MN'_f) \right] F(p) \, \text{Tr} (\bar{A}_1 A^1 + \bar{B}^1 B_1) \\
&\quad + \left[4|h_2|^2 (MN + 1) + (|h_3|^2 + |h_4|^2) MN + (h_3 \bar{h}_4 + h_4 \bar{h}_3) \right. \\
&\quad \left. + (|\alpha_2|^2 NN_f + |\alpha_4|^2 MN'_f) \right] F(p) \, \text{Tr} (\bar{A}_2 A^2 + \bar{B}^2 B_2) \\
\Pi_{bif}^{(2)}(3c) &= -\frac{1}{2} \left[\frac{N^2 + 1}{K_1^2} + \frac{M^2 + 1}{K_2^2} + \frac{4NM}{K_1 K_2} \right] F(p) \, \text{Tr} (\bar{A}_i A^i + \bar{B}^i B_i)
\end{aligned} \tag{3.6}$$

where $F(p)$ is the two-loop self-energy integral given in (A.14).

Analogously, for fundamental matter we find

$$\begin{aligned}
\Pi_{fund1}^{(2)}(3a) &= -\frac{1}{K_1^2} \left(2NM + NN_f - \frac{1}{2} (N^2 - 1) \right) F(0) \text{Tr} \left(\bar{Q}^1 Q_1 + \bar{\tilde{Q}}^1 \tilde{Q}_1 \right) \\
\Pi_{fund2}^{(2)}(3a) &= -\frac{1}{K_2^2} \left(2NM + MN'_f - \frac{1}{2} (M^2 - 1) \right) F(0) \text{Tr} \left(\bar{Q}^2 Q_2 + \bar{\tilde{Q}}^2 \tilde{Q}_2 \right) \\
\Pi_{fund1}^{(2)}(3b) &= \left[4|\lambda_1|^2 (NN_f + 1) + |\lambda_3|^2 MN'_f \right. \\
&\quad \left. + (|\alpha_1|^2 + |\alpha_2|^2) MN \right] F(p) \text{Tr} \left(\bar{Q}^1 Q_1 + \bar{\tilde{Q}}^1 \tilde{Q}_1 \right) \\
\Pi_{fund2}^{(2)}(3b) &= \left[4|\lambda_2|^2 (MN'_f + 1) + |\lambda_3|^2 NN_f \right. \\
&\quad \left. + (|\alpha_3|^2 + |\alpha_4|^2) NM \right] F(p) \text{Tr} \left(\bar{Q}^2 Q_2 + \bar{\tilde{Q}}^2 \tilde{Q}_2 \right) \\
\Pi_{fund1}^{(2)}(3c) &= -\frac{N^2 + 1}{2K_1^2} F(p) \text{Tr} \left(\bar{Q}^1 Q_1 + \bar{\tilde{Q}}^1 \tilde{Q}_1 \right) \\
\Pi_{fund2}^{(2)}(3c) &= -\frac{M^2 + 1}{2K_2^2} F(p) \text{Tr} \left(\bar{Q}^2 Q_2 + \bar{\tilde{Q}}^2 \tilde{Q}_2 \right)
\end{aligned} \tag{3.7}$$

where $F(p)$ is still given in (A.14).

We now proceed to the renormalization of the theory. We define renormalized fields as

$$\Phi = Z_\Phi^{-\frac{1}{2}} \Phi_B, \quad \bar{\Phi} = \bar{Z}_\Phi^{-\frac{1}{2}} \bar{\Phi}_B \tag{3.8}$$

where Φ stands for any chiral field of the theory, and coupling constants as

$$\begin{aligned}
h_j &= \mu^{-2\epsilon} Z_{h_j}^{-1} h_{jB} & \bar{h}_j &= \mu^{-2\epsilon} Z_{\bar{h}_j}^{-1} \bar{h}_{jB} \\
\lambda_j &= \mu^{-2\epsilon} Z_{\lambda_j}^{-1} \lambda_{jB} & \bar{\lambda}_j &= \mu^{-2\epsilon} Z_{\bar{\lambda}_j}^{-1} \bar{\lambda}_{jB} \\
\alpha_j &= \mu^{-2\epsilon} Z_{\alpha_j}^{-1} \alpha_{jB} & \bar{\alpha}_j &= \mu^{-2\epsilon} Z_{\bar{\alpha}_j}^{-1} \bar{\alpha}_{jB}
\end{aligned}$$

together with $K_1 = \mu^{2\epsilon} K_{1B}$, $K_2 = \mu^{2\epsilon} K_{2B}$. Powers of the renormalization mass μ have been introduced in order to deal with dimensionless renormalized couplings.

In order to cancel the divergences in (3.6) and (3.7) we choose

$$Z_{A^1} = Z_{\bar{A}_1} = Z_{B_1} = Z_{\bar{B}^1} = 1 - \frac{1}{64\pi^2} \left[-\frac{2NM + NN_f + 1}{K_1^2} - \frac{2NM + MN'_f + 1}{K_2^2} - \frac{2NM + 4}{K_1 K_2} + 4|h_1|^2(MN + 1) + (|h_3|^2 + |h_4|^2)MN + (h_3\bar{h}_4 + h_4\bar{h}_3) + (|\alpha_1|^2 NN_f + |\alpha_3|^2 MN'_f) \right] \frac{1}{\epsilon} \quad (3.9)$$

$$Z_{A^2} = Z_{\bar{A}_2} = Z_{B_2} = Z_{\bar{B}^2} = 1 - \frac{1}{64\pi^2} \left[-\frac{2NM + NN_f + 1}{K_1^2} - \frac{2NM + MN'_f + 1}{K_2^2} - \frac{2NM + 4}{K_1 K_2} + 4|h_2|^2(MN + 1) + (|h_3|^2 + |h_4|^2)MN + (h_3\bar{h}_4 + h_4\bar{h}_3) + (|\alpha_2|^2 NN_f + |\alpha_4|^2 MN'_f) \right] \frac{1}{\epsilon}$$

$$Z_{Q_1} = Z_{\bar{Q}^1} = Z_{\bar{Q}_1} = Z_{\bar{\bar{Q}}^1} = 1 - \frac{1}{64\pi^2} \left[-\frac{2NM + NN_f + 1}{K_1^2} + 4|\lambda_1|^2 (NN_f + 1) + |\lambda_3|^2 MN'_f + (|\alpha_1|^2 + |\alpha_2|^2)MN \right] \frac{1}{\epsilon}$$

$$Z_{Q_2} = Z_{\bar{Q}^2} = Z_{\bar{Q}_2} = Z_{\bar{\bar{Q}}^2} = 1 - \frac{1}{64\pi^2} \left[-\frac{2NM + MN'_f + 1}{K_2^2} + 4|\lambda_2|^2 (MN'_f + 1) + |\lambda_3|^2 NN_f + (|\alpha_3|^2 + |\alpha_4|^2)MN \right] \frac{1}{\epsilon}$$

Thanks to the non-renormalization theorem for the superpotential, the renormalization of the couplings is a consequence of the field renormalization. In particular, we set

$$Z_{\nu_j} = \prod_{\Phi_i} Z_{\Phi_i}^{-\frac{1}{2}} \quad (3.10)$$

where ν_j stands for any coupling of the theory and the sum is extended to all the Φ_i fields coupled by ν_j .

The anomalous dimensions and the beta-functions are given by the general prescription

$$\gamma_{\Phi_j} \equiv \frac{1}{2} \frac{\partial \log Z_{\Phi_j}}{\partial \log \mu} = -\frac{1}{2} \sum_i d_i \nu_i \frac{\partial Z_{\Phi_j}^{(1)}}{\partial \nu_i} \quad (3.11)$$

$$\beta_{\nu_j} = -d_j \nu_j^{(1)} + \sum_i \left(d_i \nu_i \frac{\partial \nu_j^{(1)}}{\partial \nu_i} \right) = \nu_j(\mu) \sum_i \gamma_i \quad (3.12)$$

where d_j is the bare dimension of the ν_j -coupling and $Z_{\Phi_j}^{(1)}$ is the coefficient of the $1/\epsilon$ pole in Z_{Φ_j} . The last equality in (3.12) follows from (3.11) and (3.10).

Reading the single pole coefficient $Z_{\Phi_j}^{(1)}$ in eqs. (3.9) we finally obtain

$$\begin{aligned}
\gamma_{A^1} = \gamma_{B_1} &= \frac{1}{32\pi^2} \left[-\frac{2NM + NN_f + 1}{K_1^2} - \frac{2NM + MN'_f + 1}{K_2^2} - \frac{2NM + 4}{K_1 K_2} \right. \\
&\quad + 4|h_1|^2(MN + 1) + (|h_3|^2 + |h_4|^2)MN + (h_3\bar{h}_4 + h_4\bar{h}_3) \\
&\quad \left. + (|\alpha_1|^2 NN_f + |\alpha_3|^2 MN'_f) \right] \\
\gamma_{A^2} = \gamma_{B_2} &= \frac{1}{32\pi^2} \left[-\frac{2NM + NN_f + 1}{K_1^2} - \frac{2NM + MN'_f + 1}{K_2^2} - \frac{2NM + 4}{K_1 K_2} \right. \\
&\quad + 4|h_2|^2(MN + 1) + (|h_3|^2 + |h_4|^2)MN + (h_3\bar{h}_4 + h_4\bar{h}_3) \\
&\quad \left. + (|\alpha_2|^2 NN_f + |\alpha_4|^2 MN'_f) \right] \\
\gamma_{Q_1} = \gamma_{\tilde{Q}_1} &= \frac{1}{32\pi^2} \left[-\frac{2NM + NN_f + 1}{K_1^2} \right. \\
&\quad \left. + 4|\lambda_1|^2(NN_f + 1) + |\lambda_3|^2 MN'_f + (|\alpha_1|^2 + |\alpha_2|^2)MN \right] \\
\gamma_{Q_2} = \gamma_{\tilde{Q}_2} &= \frac{1}{32\pi^2} \left[-\frac{2NM + MN'_f + 1}{K_2^2} \right. \\
&\quad \left. + 4|\lambda_2|^2(MN'_f + 1) + |\lambda_3|^2 NN_f + (|\alpha_3|^2 + |\alpha_4|^2)MN \right]
\end{aligned} \tag{3.13}$$

whereas the corresponding beta-functions are given by

$$\begin{aligned}
\beta_{h_1} &= 4h_1\gamma_{A^1} & \beta_{h_2} &= 4h_2\gamma_{A^2} \\
\beta_{h_3} &= 2h_3(\gamma_{A^1} + \gamma_{A^2}) & \beta_{h_4} &= 2h_4(\gamma_{A^1} + \gamma_{A^2}) \\
\beta_{\lambda_1} &= 4\lambda_1\gamma_{Q_1} & \beta_{\lambda_2} &= 4\lambda_2\gamma_{Q_2} \\
\beta_{\lambda_3} &= 2\lambda_3(\gamma_{Q_1} + \gamma_{Q_2}) \\
\beta_{\alpha_1} &= 2\alpha_1(\gamma_{A_1} + \gamma_{Q_1}) & \beta_{\alpha_2} &= 2\alpha_2(\gamma_{A_2} + \gamma_{Q_1}) \\
\beta_{\alpha_3} &= 2\alpha_3(\gamma_{A_1} + \gamma_{Q_2}) & \beta_{\alpha_4} &= 2\alpha_4(\gamma_{A_2} + \gamma_{Q_2})
\end{aligned} \tag{3.14}$$

4 The spectrum of fixed points

In this Section we study solutions to the equations $\beta_{\nu_j} = 0$ where the beta-functions are given in (3.14). We consider separately the cases with and without flavor matter.

4.1 Theories without flavors

We begin by considering the class of theories without flavors. In eqs. (3.13) we set $N_f = N'_f = 0$, $\lambda_j = \alpha_j = 0$ and solve the equations

$$\begin{aligned}
\beta_{h_1} &= 4h_1\gamma_{A^1} = 0 & \beta_{h_2} &= 4h_2\gamma_{A^2} = 0 \\
\beta_{h_3} &= 2h_3(\gamma_{A^1} + \gamma_{A^2}) = 0 & \beta_{h_4} &= 2h_4(\gamma_{A^1} + \gamma_{A^2}) = 0
\end{aligned} \tag{4.1}$$

When $h_j \neq 0$ for any j the conditions (4.1) are equivalent to $\gamma_{A^1} = \gamma_{A^2} = 0$, that is no UV divergences appear at two-loops. On the other hand, if we restrict to the surface $h_1 = h_2 = 0$, the beta-functions are zero when $\gamma_{A^1} + \gamma_{A^2} = 0$, which in principle would not require the anomalous dimensions to vanish. However, it is easy to see from (3.13) that for $h_1 = h_2 = 0$ we have $\gamma_{A^1} = \gamma_{A^2}$ and again $\beta_{h_3} = \beta_{h_4} = 0$ imply the vanishing of all the anomalous dimensions. Therefore, at two loops the request for vanishing beta-functions is equivalent to the request of finiteness.

We first study the class of theories with $h_1 = h_2 = 0$. In this case we find convenient to redefine the couplings as [52]

$$y_1 = h_3 + h_4 \quad , \quad y_2 = h_3 - h_4 \quad (4.2)$$

In fact, writing the superpotential in terms of the new couplings

$$\int d^4x d^2\theta \left[\frac{y_1}{2} \text{Tr}(A^1 B_1 A^2 B_2 + A^2 B_1 A^1 B_2) + \frac{y_2}{4} \epsilon_{ij} \epsilon^{kl} \text{Tr}(A^i B_k A^j B_l) \right] \quad (4.3)$$

it is easy to see that y_1 is associated to a $SU(2)_A \times SU(2)_B$ breaking perturbation, whereas y_2 is symmetry preserving.

For real couplings, the anomalous dimensions vanish when

$$y_1^2(MN + 1) + y_2^2(MN - 1) = 2(2MN + 1) \left(\frac{1}{K_1^2} + \frac{1}{K_2^2} \right) + 2 \frac{2MN + 4}{K_1 K_2} \quad (4.4)$$

This describes an ellipse in the parameter space. For $K_{1,2}$ sufficiently large it is very closed to the origin and solutions fall in the perturbative regime. The ellipse degenerates to a circle in the large M, N limit. Fixed points corresponding to $y_1 \neq 0$ ($h_4 \neq -h_3$) describe $\mathcal{N} = 2$ superconformal theories with $U(1)_A \times U(1)_B$ global symmetry (2.8).

A more symmetric conformal point is obtained by solving (4.4) under the condition $y_1 = 0$. The solution

$$h_3 = -h_4 = \sqrt{\frac{2MN + 1}{2(MN - 1)} \left(\frac{1}{K_1^2} + \frac{1}{K_2^2} \right) + \frac{MN + 2}{MN - 1} \frac{1}{K_1 K_2}} \quad (4.5)$$

corresponds to a superconformal theory with $SU(2)_A \times SU(2)_B$ global symmetry. This is the theory conjectured in [42]. When $K_1 = -K_2 \equiv K$ it reduces to $h_3 = -h_4 = 1/K$ and we recover the $\mathcal{N} = 6$ ABJ model [34] and, for $N = M$, the ABJM one [8].

More generally, we study fixed points with $h_j \neq 0$ for any j . In this case we have two equations, $\gamma_{A^1} = \gamma_{A^2} = 0$, for four unknowns. The spectrum of fixed points then spans a two dimensional surface which for real couplings is given by

$$h_1^2 = h_2^2 = \frac{1}{4(MN + 1)} \left[(2MN + 1) \left(\frac{1}{K_1^2} + \frac{1}{K_2^2} \right) + 2 \frac{MN + 2}{K_1 K_2} - MN(h_3^2 + h_4^2) - 2h_3 h_4 \right] \quad (4.6)$$

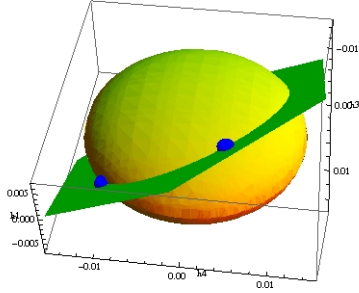


Figure 4: The exactly marginal surface of fixed points in the space of h_i couplings, restricted to the subspace $h_1 = h_2$. The parameters have been chosen as $K_1 = 150, K_2 = 237, N = 43, M = 30$. The dots denote the $\mathcal{N} = 3$ and the $\mathcal{N} = 2, SU(2)_A \times SU(2)_B$ fixed points belonging to the ellipsoid. The plane represents the class of theories (2.11) with $SU(2)$ global symmetry and its intersection with the ellipsoid is the line described by (4.8).

This equation describes an ellipsoid in the four dimensional h -space as given in Fig. 4, localized in the subspace $h_1 = h_2$ (or equivalently $h_1 = -h_2$). A particular point on this surface corresponds to $h_3 = 1/K_1$ and $h_4 = 1/K_2$ with, consequently, $h_1 = h_2 = \frac{1}{2}(\frac{1}{K_1} + \frac{1}{K_2})$. This is the $\mathcal{N} = 3$ superconformal theory discussed in [42]⁵.

The locus $h_1 = h_2 = 0, h_3 = -h_4$ of this surface is the $\mathcal{N} = 2, SU(2)_A \times SU(2)_B$ invariant superconformal theory (4.5). Therefore, the $\mathcal{N} = 3$ and the $\mathcal{N} = 2, SU(2)_A \times SU(2)_B$ superconformal points are continuously connected by the surface (4.6).

We can select a particular line of fixed points interpolating between the two theories, by setting

$$h_1 = h_2 = \frac{1}{2}(h_3 + h_4) \quad (4.7)$$

and, consequently

$$h_3^2 + h_4^2 + 2\frac{MN+2}{2MN+1}h_3h_4 = \frac{1}{K_1^2} + \frac{1}{K_2^2} + 2\frac{MN+2}{K_1K_2(2MN+1)} \quad (4.8)$$

These are $SU(2)$ invariant, $\mathcal{N} = 2$ superconformal theories with superpotential (2.11). The existence of a line of $SU(2)$ invariant fixed points interpolating between the two theories was already conjectured in [42].

So far we have considered real solutions to the equations $\beta_{\nu_j} = 0$. We now discuss the case of complex couplings focusing in particular on the so-called β -deformations.

In the class of theories with $h_1 = h_2 = 0$ we look for solutions of the form

$$h_3 = he^{i\pi\beta} \quad , \quad h_4 = -he^{-i\pi\beta} \quad (4.9)$$

⁵Finiteness properties of $\mathcal{N} = 3$ CS-matter theories have also been discussed in [43] within the $\mathcal{N} = 3$ harmonic superspace setup.

which implies $y_1 = 2h \sin \pi\beta$, $y_2 = 2h \cos \pi\beta$ in (4.2). The condition for vanishing beta-functions then reads

$$h^2 MN - h^2 \cos 2\pi\beta = \frac{1}{2}(2MN + 1) \left(\frac{1}{K_1^2} + \frac{1}{K_2^2} \right) + \frac{MN + 2}{K_1 K_2} \quad (4.10)$$

This describes a line of fixed points which correspond to superconformal beta-deformations of the $SU(2)_A \times SU(2)_B$ invariant theory (4.5). For $\beta \neq 0$ the global symmetry is broken to $U(1)_A \times U(1)_B$ in (2.8) and the deformed theory is only $\mathcal{N} = 2$ supersymmetric. In particular, setting $K_1 = -K_2$ we obtain the β -deformed ABJM/ABJ theories studied in [40].

In the large M, N limit the β -dependence of equation (4.10) disappears, consistently with the fact that in planar Feynman diagrams the effects of the deformation are invisible [58]. In this limit the condition for superconformal invariance reads

$$h^2 = \frac{1}{K_1^2} + \frac{1}{K_2^2} + \frac{1}{K_1 K_2} \quad (4.11)$$

which reduces to $h = 1/K$ for opposite CS levels.

The analysis of β -deformations can be extended to theories with $h_1, h_2 \neq 0$. Since they enter the anomalous dimensions only through $|h_1|^2$ and $|h_2|^2$ we can take them to be arbitrarily complex and still make the ansatz (4.9) for h_3, h_4 . The surface of fixed points is then given by

$$\begin{aligned} |h_1|^2 = |h_2|^2 = & \frac{1}{4(MN + 1)} \left[(2MN + 1) \left(\frac{1}{K_1^2} + \frac{1}{K_2^2} \right) + 2 \frac{MN + 2}{K_1 K_2} \right. \\ & \left. - 2h^2 MN + 2h^2 \cos 2\pi\beta \right] \end{aligned} \quad (4.12)$$

and describes superconformal β -deformations of $\mathcal{N} = 2$ invariant theories.

The results of this Section agree with the ones in [57] obtained by using the three-algebra formalism.

4.2 Theories with flavors

As in the previous case, when all the couplings are non-vanishing, the request for zero beta-functions implies the finiteness conditions $\gamma_{\Phi_i} = 0$. These provide four constraints on a set of eleven unknowns (see eqs. (3.13)). Therefore, in the space of the coupling constants the spectrum of fixed points spans a seven dimensional hypersurface given by the equations

$$\begin{aligned} |\alpha_2|^2 = & \frac{1}{NN_f K_1^2 K_2^2} \left\{ K_2^2 (2NM + NN_f + 1) + K_1^2 (2NM + MN'_f + 1) \right. \\ & + 2K_1 K_2 (NM + 2) - 4|h_2|^2 K_1^2 K_2^2 (MN + 1) \\ & \left. - K_1^2 K_2^2 [(|h_3|^2 + |h_4|^2)MN + (h_3 \bar{h}_4 + h_4 \bar{h}_3) + |\alpha_4|^2 MN'_f] \right\} \end{aligned}$$

$$\begin{aligned}
|\alpha_3|^2 &= \frac{1}{MN'_f K_1^2 K_2^2} \left\{ K_2^2 (2NM + NN_f + 1) + K_1^2 (2NM + MN'_f + 1) \right. \\
&\quad + 2K_1 K_2 (NM + 2) - 4|h_1|^2 K_1^2 K_2^2 (MN + 1) \\
&\quad \left. - K_1^2 K_2^2 [(|h_3|^2 + |h_4|^2)MN + (h_3 \bar{h}_4 + h_4 \bar{h}_3) + |\alpha_1|^2 NN_f] \right\} \\
|\lambda_1|^2 &= \frac{1}{4(NN_f + 1)K_1^2} \left\{ 2NM + NN_f + 1 - K_1^2 [|\lambda_3|^2 MN'_f + (|\alpha_1|^2 + |\alpha_2|^2)MN] \right\} \\
|\lambda_2|^2 &= \frac{1}{4(MN'_f + 1)K_2^2} \left\{ 2NM + MN'_f + 1 - K_2^2 [|\lambda_3|^2 NN_f + (|\alpha_3|^2 + |\alpha_4|^2)MN] \right\}
\end{aligned} \tag{4.13}$$

When $K_1 = -K_2 \equiv K$ a particular point on this surface corresponds to

$$\begin{aligned}
h_1 = h_2 = 0 \quad , \quad h_3 = -h_4 = \frac{1}{K} \\
\lambda_1 = -\lambda_2 = \frac{1}{2K} \quad , \quad \lambda_3 = 0 \\
\alpha_1 = \alpha_2 = \frac{1}{K} \quad , \quad \alpha_3 = \alpha_4 = -\frac{1}{K}
\end{aligned} \tag{4.14}$$

and describes the $\mathcal{N} = 3$ ABJ/ABJM models with flavor matter [44, 45, 46].

More generally, allowing $K_2 \neq -K_1$ we find the fixed point

$$\begin{aligned}
h_1 = h_2 = \frac{1}{2} \left(\frac{1}{K_1} + \frac{1}{K_2} \right) \quad , \quad h_3 = \frac{1}{K_1} \quad , \quad h_4 = \frac{1}{K_2} \\
\lambda_1 = \frac{1}{2K_1} \quad , \quad \lambda_2 = \frac{1}{2K_2} \quad , \quad \lambda_3 = 0 \\
\alpha_1 = \alpha_2 = \frac{1}{K_1} \quad , \quad \alpha_3 = \alpha_4 = \frac{1}{K_2}
\end{aligned} \tag{4.15}$$

which corresponds to a superconformal theory obtained from the $\mathcal{N} = 3$ theory of [42] by the addition of flavor matter [45]. The superpotential

$$\begin{aligned}
\mathcal{S}_{\text{pot}} &= \int d^3x d^2\theta \text{Tr} \left\{ \frac{1}{2} \left(\frac{1}{K_1} + \frac{1}{K_2} \right) [(A^1 B_1)^2 + (A^2 B_2)^2] \right. \\
&\quad + \frac{1}{K_1} (A^1 B_1 A^2 B_2) + \frac{1}{K_2} (A^2 B_1 A^1 B_2) + \frac{1}{2K_1} (Q_1 \tilde{Q}_1)^2 + \frac{1}{2K_2} (Q_2 \tilde{Q}_2)^2 \\
&\quad \left. + \frac{1}{K_1} [\tilde{Q}_1 A^i B_i Q_1] + \frac{1}{K_2} [\tilde{Q}_2 B_i A^i Q_2] \right\} + h.c.
\end{aligned} \tag{4.16}$$

can be thought of as arising from the action

$$\begin{aligned}
\mathcal{S} &= \mathcal{S}_{\text{CS}} + \mathcal{S}_{\text{mat}} \\
&+ \int d^3x d^2\theta \left[-\frac{K_1}{2} \text{Tr}(\Phi_1^2) + \text{Tr}(B_i \Phi_1 A^i) + \text{Tr}(\tilde{Q}_1 \Phi_1 Q_1) \right] \\
&+ \int d^3x d^2\theta \left[-\frac{K_2}{2} \text{Tr}(\Phi_2^2) + \text{Tr}(A^i \Phi_2 B_i) + \text{Tr}(\tilde{Q}_2 \Phi_2 Q_2) \right] + h.c. \quad (4.17)
\end{aligned}$$

after integration on the Φ_1, Φ_2 chiral superfields belonging to the adjoint representations of the two gauge groups and giving the $\mathcal{N} = 4$ completion of the vector multiplet. Therefore, as in the unflavored case, the theory exhibits $\mathcal{N} = 3$ supersymmetry with the couples $(A, B^\dagger)_i, (Q, \tilde{Q}^\dagger)_1^r$ and $(Q, \tilde{Q}^\dagger)_2^{r'}$ realizing $(2 + N_f + N'_f)$ $\mathcal{N} = 4$ hypermultiplets (The CS terms break $\mathcal{N} = 4$ to $\mathcal{N} = 3$).

As already discussed, in the absence of flavors the $\mathcal{N} = 3$ superconformal theory is connected by the line of fixed points (4.8) to a $\mathcal{N} = 2$, $SU(2)_A \times SU(2)_B$ invariant theory. We now investigate whether a similar pattern arises even when flavors are present.

To this end, we first choose

$$h_1 = h_2 = \frac{1}{2}(h_3 + h_4) \quad , \quad \alpha_1 = \alpha_2 \quad , \quad \alpha_3 = \alpha_4 \quad (4.18)$$

with λ_j arbitrary. This describes a set of $\mathcal{N} = 2$ theories with global $SU(2)$ invariance in the bifundamental sector.

Solving the equations $\beta_{\nu_j} = 0$ for real couplings we find a whole line of $SU(2)_A \times SU(2)_B$ invariant fixed points parametrized by the unconstrained coupling λ_3

$$\begin{aligned}
\alpha_1 = \alpha_3 &= 0 \\
h_3 = -h_4 &= \sqrt{\frac{(2NM + MN'_f + 1)K_1^2 + 2(MN + 2)K_1K_2 + (2MN + NN_f + 1)K_2^2}{2(MN - 1)K_1^2K_2^2}} \\
\lambda_1^2 &= \frac{2MN + NN_f + 1 - K_1^2MN'_f\lambda_3^2}{4K_1^2(NN_f + 1)} \\
\lambda_2^2 &= \frac{2MN + MN'_f + 1 - K_2^2NN_f\lambda_3^2}{4K_2^2(MN'_f + 1)} \quad (4.19)
\end{aligned}$$

A four dimensional hypersurface of $\mathcal{N} = 2$ fixed points given by

$$\begin{aligned}
\alpha_1^2 &= \frac{1}{2MNK_1^2} \left[-4K_1^2(NN_f + 1)\lambda_1^2 + NN_f + 2MN + 1 - MN'_f\lambda_3^2K_1^2 \right] \\
\alpha_3^2 &= \frac{1}{2MNK_2^2} \left[-4K_2^2(MN'_f + 1)\lambda_2^2 + MN'_f + 2MN + 1 - NN_f\lambda_3^2K_2^2 \right] \\
h_3 &= -\frac{1}{2MN + 1} \left\{ (MN + 2)h_4 \pm \left[(2MN + 1) \left(MN'_f(-\alpha_3^2 + 4\lambda_2^2 + \lambda_3^2) \right. \right. \right. \\
&\quad \left. \left. \left. + 2NM(\alpha_1^2 + \alpha_3^2 + 1) + NN_f(-\alpha_1^2 + 4\lambda_1^2 + \lambda_3^2) \right. \right. \right. \\
&\quad \left. \left. \left. + 4(\lambda_1^2 + \lambda_2^2) + \frac{4}{K_1K_2} \right) - 3h_4^2(M^2N^2 - 1) \right]^{1/2} \right\}
\end{aligned} \tag{4.20}$$

connects the line of $\mathcal{N} = 2$, $SU(2)_A \times SU(2)_B$ invariant theories (4.19) to the $\mathcal{N} = 3$ theory (4.15). This is the analogous of the fixed line (4.8) found in the unflavored theories.

Before closing this Section we address the question of superconformal invariance versus finiteness for theories with flavor matter. In the bifundamental sector, the only possibility to have vanishing beta-functions without vanishing anomalous dimensions is by setting $h_1 = h_2 = 0$. When flavor matter is present, this does not necessarily imply $\gamma_{A^1} = \gamma_{A^2}$, so we can solve for $\beta_{h_3, h_4} = \gamma_{A^1} + \gamma_{A^2} = 0$ without requiring the two γ 's to vanish separately. Once these equations have been solved in the bifundamental sector, in the flavor sector we choose $\lambda_1 = \lambda_2 = 0$ and $\alpha_1 = \alpha_4 = 0$ (or equivalently, $\alpha_2 = \alpha_3 = 0$) in order to avoid $\gamma_{Q_1} = \gamma_{Q_2} = 0$. We are then left with five couplings subject to the three equations $\gamma_{A^1} + \gamma_{A^2} = 0$, $\gamma_{A^1} + \gamma_{Q_2} = 0$ and $\gamma_{A^2} + \gamma_{Q_1} = 0$. Solutions correspond to superconformal but *not finite* theories. We note that this is true as long as we work with M, N finite. In the large M, N limit with $N_f, N'_f \ll M, N$ we are back to $\gamma_{A^1} = \gamma_{A^2}$, as flavor contributions are subleading. In this case superconformal invariance requires finiteness.

5 Infrared stability

We now study the RG flows around the fixed points of main interest in order to establish whether they are IR attractors or repulsors. In particular, we concentrate on the ABJ/ABJM theories, $\mathcal{N} = 3$ and $SU(2)_A \times SU(2)_B$ $\mathcal{N} = 2$ superconformal points, in all cases with and without flavors.

The behavior of the system around a given fixed point ν_0 is determined by studying the stability matrix

$$\mathcal{M}_{ij} \equiv \frac{d\beta_i}{d\nu_j}(\nu_0) \tag{5.1}$$

Diagonalizing \mathcal{M} , positive eigenvalues correspond to directions of increasing β -functions, whereas negative eigenvalues give decreasing betas. It follows that the fixed point is IR stable if \mathcal{M} has all positive eigenvalues, whereas negative eigenvalues represent directions where a classically marginal operator becomes relevant.

If null eigenvalues are present we need compute derivatives of the stability matrix along the directions individuated by the corresponding eigenvectors. If along a null direction the second derivative of the beta-function is different from zero, then the function has a parabolic behavior and the system is unstable.

We apply these criteria to the two-loop beta-functions (3.14).

5.1 Theories without flavors

We begin with the $\mathcal{N} = 2$ theories without flavor discussed in Section 4.1. As shown, the nontrivial fixed points lie on a two dimensional ellipsoid and particular points on it are the $\mathcal{N} = 3$ and the $\mathcal{N} = 2$ $SU(2)_A \times SU(2)_B$ invariant theories. Since the ellipsoid is localized in the subspaces $h_1 = \pm h_2$ we restrict our discussion to the $h_1 = h_2$ case.

When $K_1 = -K_2 \equiv K$, the set of theories with $h_1 = h_2 = 0$ has been already studied in [52]. In this case the ellipsoid reduces to an ellipse in the (y_1, y_2) plane with y_1 and y_2 defined in eq. (4.2). It has been shown that at the order we are working the RG trajectories are straight lines passing through the origin and intersecting the ellipse. Infrared flows point towards the ellipse so proving that the whole line of fixed points is IR stable. However, every single point has only one direction of stability which corresponds to the RG trajectory passing through it. When perturbed along any other direction the system flows to a different fixed point on the curve. In the ABJ/ABJM case the direction of stability is described by $SU(2)_A \times SU(2)_B$ preserving perturbations.

This can be understood by computing the stability matrix at $h_1 = h_2 = 0$, $h_3 = -h_4 = 1/K$ and diagonalizing it. We find that mutual orthogonal directions are $(h_1 = h_2, y_1, y_2)$ and the corresponding eigenvalues are

$$\mathcal{M} = \text{diag}\left\{0, 0, \frac{MN - 1}{2\pi^2 K^2}\right\} \quad (5.2)$$

For $M, N > 1$ the third eigenvalue is positive, so the ABJ/ABJM theory is an attractor along the y_2 -direction.

Solving the degeneracy of null eigenvalues requires computing the matrix of second derivatives. In particular, looking at the y_1 -direction we find

$$\frac{\partial^2 \beta_{y_1}}{\partial y_1^2} = \frac{1 - MN}{2\pi^2 K} \quad (5.3)$$

Since it is non-vanishing, the y_1 coordinate is a line of instability. Therefore, when perturbed by a $SU(2)_A \times SU(2)_B$ violating operator the system leaves the ABJM fixed point and flows to a less symmetric fixed point along a RG trajectory.

We now generalize the analysis to the case of different CS levels. In this case we refer to the surface of fixed points in Fig. 5 where for clearness only half of the ellipsoid has been drawn. The black line corresponds to $\mathcal{N} = 2$ superconformal theories with $h_1 = h_2 = 0$, where the green point is the $SU(2)_A \times SU(2)_B$ invariant model. The red point is instead the $\mathcal{N} = 3$ superconformal theory.

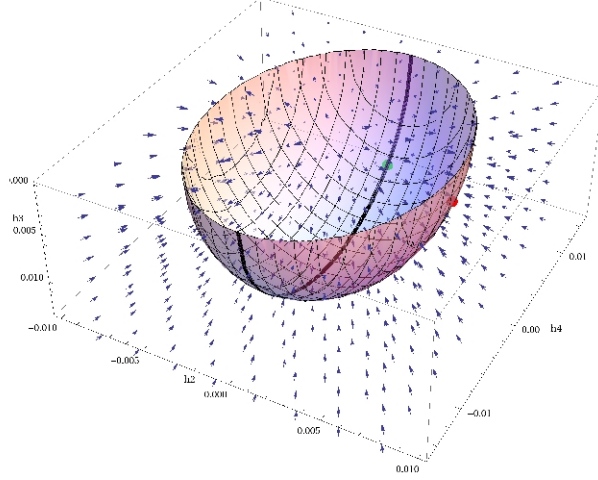


Figure 5: The ellipsoid of fixed points and the RG flows for $\mathcal{N} = 2$ theories in the space of couplings $(h_1 = h_2, h_3, h_4)$. Arrows point towards IR directions. The parameters are $K_1 = 150$, $K_2 = 237$, $M = 30$ and $N = 43$.

From eq. (3.13) we see that in the $h_1 = h_2$ subsector we have $\gamma_{A^1} = \gamma_{A^2}$. As a consequence, all the beta-functions are equal and the RG flow equations simplify to

$$\frac{dh_i}{dh_j} = \frac{h_i}{h_j} \quad (5.4)$$

In the three dimensional parameter space $(h_1 = h_2, h_3, h_4)$, solutions are all the straight lines passing through the origin and intersecting the ellipsoid.

Infrared flows can be easily studied by plotting the vector $(-\beta_{h_1}, -\beta_{h_3}, -\beta_{h_4})$ in each point. The result is given in Fig. 5 where it is clear that the entire surface is globally IR stable.

In order to study the local behavior of the system in proximity of a given fixed point, we compute the stability matrix at the point (4.6) and diagonalize it. Surprisingly, the eigenvalues turn out to be independent of the particular point on the surface

$$\mathcal{M} = \text{diag} \left\{ 0, 0, \frac{K_1^2 + 4K_1K_2 + K_2^2 + 2(K_1^2 + K_1K_2 + K_2^2)MN}{4K_1^2K_2^2\pi^2} \right\} \quad (5.5)$$

The two null eigenvalues characterize directions of instability. In fact, we can solve the degeneracy by computing the matrix of second derivatives respect to the corresponding eigenvectors. It turns out that in all cases the beta functions have a parabolic behavior along those directions and the system is unstable.

For example, at the $\mathcal{N} = 2$, $SU(2)_A \times SU(2)_B$ invariant fixed point (green dot in the Figure), these eigenvectors are $\{0, 1, 1\}$ and $\{1, 0, 0\}$, which are precisely the directions

$h_3 = h_4$ and h_1 , tangent to the surface at that point. It is clear from Fig. 5 that if we perturb the system along these directions it will intercept a RG trajectory which leads it to another fixed point.

The stability properties of the β -deformed theories are easily inferred from the previous discussion. In fact, performing the following rotation of the couplings

$$h \cos(\pi\beta) = \frac{x}{2}, \quad h \sin(\pi\beta) = \frac{y}{2} \quad (5.6)$$

the condition (4.10) for vanishing beta-functions becomes

$$\frac{1}{4}(MN-1)x^2 + \frac{1}{4}(MN+1)y^2 = \frac{1}{2}(2MN+1) \left(\frac{1}{K_1^2} + \frac{1}{K_2^2} \right) + \frac{MN+2}{K_1 K_2} \quad (5.7)$$

This is exactly the ellipse (4.4) of the undeformed case. Therefore, the infrared stability properties of this curve are precisely the ones discussed before.

5.2 Theories with flavors

We now turn to flavored theories introduced in Section 4.2. The form of the stability matrix is quite cumbersome, but we can analyze the effects of the interactions with flavor multiplets by studying particular examples.

As the simplest case we consider the class of theories described by the superpotential (2.4) where only λ_i couplings have been turned on. The β -functions of the theory split into two completely decoupled sectors: The former is the four dimensional space of couplings h_1, h_2, h_3, h_4 , whose stability was addressed in the previous subsection; the latter is the three dimensional space of λ_i couplings.

Looking at the λ_i sector, nontrivial solutions to $\beta_i = 0$ describe a curve of fixed points given by expressing λ_1 and λ_2 as functions of λ_3 (see eqs. (4.13)). It is the two-branch curve of Fig. 6. The most general solution includes also isolated points where either λ_1 or λ_2 vanish.

Drawing the vector $(-\beta_{\lambda_1}, -\beta_{\lambda_2}, -\beta_{\lambda_3})$ in each point of the parameter space we obtain the RG flow configurations as given in Fig. 6. It is then easy to see that the isolated fixed points are always unstable since the RG flows drive the theory to one of the two branches in the IR.

This behavior can be also inferred from the structure of the stability matrix. In fact, one can check that when evaluated on the curve the matrix has two positive eigenvalues, whereas when evaluated at the isolated solutions it has negative eigenvalues. As before, theories living on the curve have directions of local instability signaled by the presence of a null eigenvalue which can be solved at second order in the derivatives. The direction of instability is tangent to the curve.

Finally, we consider the more complicated case of theories with superpotential (2.4) where only the α_i couplings are non-vanishing. This time the β -functions for the h_i sector do not decouple from the β -functions of the α_i sector and the analysis of fixed points becomes quite complicated.

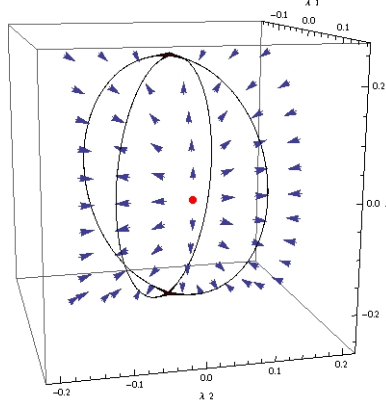


Figure 6: A sketch of the RG flow for the λ_i couplings only. A curve of fixed points is shown, which is IR stable. The red dot represents an isolated IR unstable fixed point. Here, the parameters are $K_1 = K_2 = 20$, $M = N = 10$, $N_f = N'_f = 1$.

In order to effort the calculation we restrict to the class of $U(N) \times U(N)$ theories (therefore $N_f = N'_f$) with $|K_1| = |K_2|$. This allows to choose α_i all equal to α . Moreover, we set $h_1 = h_2$ and $y_1 = 0$ in (4.2). The spectrum of fixed points and the RG trajectories are then studied in the three-dimensional space of parameters (α, h_1, y_2) .

The β -functions vanish for vanishing couplings (free theory) and for $\gamma_{A^1} = \gamma_{Q_1} = 0$. Nontrivial solutions for α are obtained from $\gamma_{Q_1} = 0$. Using eqs. (3.13), for real couplings we find

$$\alpha = \pm \sqrt{\frac{2N^2 + NN_f + 1}{2N^2 K_1^2}} \quad (5.8)$$

Fixing α to be one of the three critical values (zero or one of these two values) we can solve $\gamma_{A^1} = 0$. As in the previous cases this describes an ellipse on the (h_1, y_2) plane localized at $\alpha = \text{const}$. For theories with $K_1 = K_2$ the configuration of fixed points is given in Fig. 7 where we have chosen to draw only half ellipses.

Renormalization group flows are obtained by plotting the vector $(-\beta_\alpha, -\beta_{h_1}, -\beta_{y_2})$. The stability of fixed points is better understood by projecting RG trajectories on orthogonal planes. Looking for instance at the $h_1 = 0$ plane we obtain the configurations in Fig. 8 where the red dots indicate the origin and the intersections of the three ellipses with the plane.

From this picture we immediately infer that the free theory is an IR unstable fixed point since the system is always driven towards nontrivial fixed points. Among them, the ones corresponding to $\alpha \neq 0$ are attractors, whereas $\alpha = 0$ does not seem to be a preferable point for the theory. In fact, it is reached flowing along the $\alpha = 0$ trajectory, but as soon as we perturb the system with a marginal operator corresponding to $\alpha \neq 0$ it will flow to one of the two nontrivial points. We conclude that if we add flavor degrees of freedom the system requires a nontrivial interaction with bifundamental matter in order to reach a stable superconformal configuration in the infrared region.

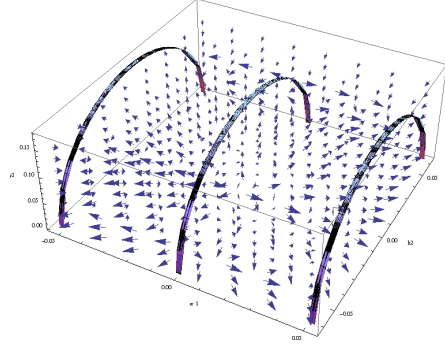


Figure 7: The three ellipses of fixed points and the RG flows for $\mathcal{N} = 2$ theories with α couplings turned on. Arrows point towards IR directions. The parameters are $K_1 = K_2 = 20$, $M = N = 10$, $N_f = N'_f = 1$.

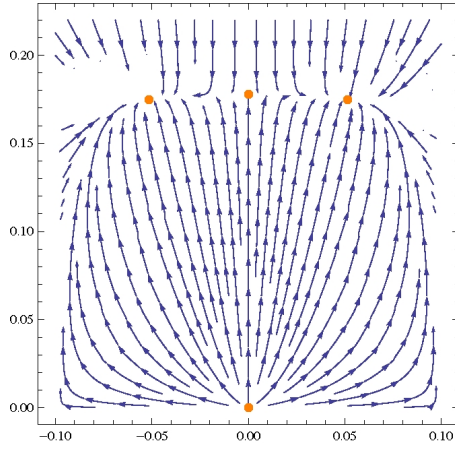


Figure 8: RG trajectories on the $h_1 = 0$ plane for $\mathcal{N} = 2$ theories with α couplings turned on. Arrows point towards IR directions. The parameters are $K_1 = K_2 = 20$, $M = N = 10$, $N_f = N'_f = 1$.

6 Conclusions

In this paper we have investigated the spectrum of superconformal fixed points of a large class of $\mathcal{N} = 2$, $U(N) \times U(M)$ Chern–Simons theories with bifundamental matter, flavored and not flavored.

We have quantized the theories in a manifest $\mathcal{N} = 2$ superspace setup and evaluated the beta-functions perturbatively, at the first nontrivial order. We have then determined the whole spectrum of fixed points by setting the betas to zero, studied the RG flows and the stability properties of the solutions. Choosing the CS levels sufficiently large compared to the rank of the gauge groups all the solutions fall inside the perturbative regime and we are allowed to investigate the IR dynamics around fixed points perturbatively.

In all cases we have found compact surfaces of fixed points. They contain as non-isolated solutions, points corresponding to $\mathcal{N} = 6$ ABJ/ABJM theories with and without flavors and theories corresponding to turning on a Romans mass in the dual supergravity description.

In a neighborhood of the surface the RG trajectories are straight lines intersecting it. Infrared RG flows always point towards the surface which is then globally stable. However, a local instability is present and can be understood as follows. Around a given fixed point the addition of a non-exactly marginal operator drives the system out of the fixed point. If it happens along the RG trajectory which intersects the surface at that point the system will flow back to the original superconformal point. But if this does not happen, the perturbed system meets another RG trajectory which leads it to an infinitesimally closed but different fixed point on the surface.

In particular, the ABJM fixed point is stable only respect to $SU(2) \times SU(2)$ invariant perturbations. Any other perturbation drives the system to a less symmetric superconformal point.

When interacting flavors are present, it comes out that IR stability is favored by a non-trivial interaction with the bifundamental matter other than with the gauge fields.

Our analysis could be extended to different classes of theories for which a perturbative investigation of the IR region makes sense. One example is the class of $\mathcal{N} = 1$ theories introduced in [42] and corresponding to splitting the two CS levels in the ABJ/ABJM action written in $\mathcal{N} = 1$ superspace formalism.

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A Notations and conventions

In this Appendix we list a number of conventions which are useful for understanding the technical part of the paper.

The world-volume metric is $g^{\mu\nu} = \text{diag}(-1, +1, +1)$ with index range $\mu = 0, 1, 2$. We use Dirac matrices $(\gamma^\mu)_\alpha{}^\beta = (i\sigma^2, \sigma^1, \sigma^3)$ satisfying $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \epsilon^{\mu\nu\rho}\gamma_\rho$.

The fermionic coordinates of $\mathcal{N} = 2$ superspace are two real two-component spinors $\theta_i, i = 1, 2$ which we combine into a complex two spinor

$$\theta^\alpha = \frac{1}{\sqrt{2}}(\theta_1^\alpha + i\theta_2^\alpha) \quad , \quad \bar{\theta}^\alpha = \frac{1}{\sqrt{2}}(\theta_1^\alpha - i\theta_2^\alpha) \quad (\text{A.1})$$

Indices are raised and lowered according to $\theta^\alpha = C^{\alpha\beta}\theta_\beta$, $\theta_\alpha = \theta^\beta C_{\beta\alpha}$, with $C^{12} = -C_{12} = i$. We have

$$\theta_\alpha\theta_\beta = C_{\beta\alpha}\theta^2 \quad , \quad \theta^\alpha\theta^\beta = C^{\beta\alpha}\theta^2 \quad (\text{A.2})$$

and likewise for $\bar{\theta}$ and derivatives.

Supercovariant derivatives and susy generators are

$$\begin{aligned} D_\alpha &= \partial_\alpha + \frac{i}{2}\bar{\theta}^\beta\partial_{\alpha\beta} = \frac{1}{\sqrt{2}}(D_\alpha^1 - iD_\alpha^2) \quad , \quad \bar{D}_\alpha = \bar{\partial}_\alpha + \frac{i}{2}\theta^\beta\partial_{\alpha\beta} = \frac{1}{\sqrt{2}}(D_\alpha^1 + iD_\alpha^2) \\ Q_\alpha &= i(\partial_\alpha - \frac{i}{2}\bar{\theta}^\beta\partial_{\alpha\beta}) \quad , \quad \bar{Q}_\alpha = i(\bar{\partial}_\alpha - \frac{i}{2}\theta^\beta\partial_{\alpha\beta}) \end{aligned} \quad (\text{A.3})$$

with the only non-trivial anti-commutators

$$\{D_\alpha, \bar{D}_\beta\} = i\partial_{\alpha\beta} \quad , \quad \{Q_\alpha, \bar{Q}_\beta\} = i\partial_{\alpha\beta} \quad (\text{A.4})$$

We use the following conventions for integration

$$d^2\theta \equiv \frac{1}{2}d\theta^\alpha d\theta_\alpha \quad , \quad d^2\bar{\theta} \equiv \frac{1}{2}d\bar{\theta}^\alpha d\bar{\theta}_\alpha \quad , \quad d^4\theta \equiv d^2\theta d^2\bar{\theta} \quad (\text{A.5})$$

such that

$$\int d^2\theta \theta^2 = -1 \quad \int d^2\bar{\theta} \bar{\theta}^2 = -1 \quad \int d^4\theta \theta^2 \bar{\theta}^2 = 1 \quad (\text{A.6})$$

The components of a chiral and an anti-chiral superfield, $Z(x_L, \theta)$ and $\bar{Z}(x_R, \bar{\theta})$, are a complex boson ϕ , a complex two-component fermion ψ and a complex auxiliary scalar F . Their component expansions are given by

$$\begin{aligned} Z &= \phi(x_L) + \theta^\alpha\psi_\alpha(x_L) - \theta^2 F(x_L) \\ \bar{Z} &= \bar{\phi}(x_R) + \bar{\theta}^\alpha\bar{\psi}_\alpha(x_R) - \bar{\theta}^2 \bar{F}(x_R) \end{aligned} \quad (\text{A.7})$$

where

$$\begin{aligned} x_L^\mu &= x^\mu + i\theta\gamma^\mu\bar{\theta} \\ x_R^\mu &= x^\mu - i\theta\gamma^\mu\bar{\theta} \end{aligned} \quad (\text{A.8})$$

The components of the vector superfield $V(x, \theta, \bar{\theta})$ in Wess-Zumino gauge ($V| = D_\alpha V| = D^2 V| = 0$) are the gauge field $A_{\alpha\beta}$, a complex two-component fermion λ_α , a real scalar σ and an auxiliary scalar D , such that

$$V = i \theta^\alpha \bar{\theta}_\alpha \sigma(x) + \theta^\alpha \bar{\theta}^\beta A_{\alpha\beta}(x) - \theta^2 \bar{\theta}^\alpha \bar{\lambda}_\alpha(x) - \bar{\theta}^2 \theta^\alpha \lambda_\alpha(x) + \theta^2 \bar{\theta}^2 D(x). \quad (\text{A.9})$$

For $SU(N)$ we use the $N \times N$ hermitian matrix generators T^a ($a = 1, \dots, N^2 - 1$) and for $U(N)$

$$T^A = (T^0, T^a) \quad \text{with} \quad T^0 = \frac{1}{\sqrt{N}} \quad (\text{A.10})$$

The generators are normalized as $\text{Tr} T^A T^B = \delta^{AB}$.

Completeness implies

$$\begin{aligned} U(N) : \quad & \text{Tr} A T^A \text{Tr} B T^A = \text{Tr} A B \quad , \quad \text{Tr} A T^A B T^A = \text{Tr} A \text{Tr} B \\ SU(N) : \quad & \text{Tr} A T^a \text{Tr} B T^a = \text{Tr} A B - \frac{1}{N} \text{Tr} A \text{Tr} B \\ & \text{Tr} A T^a B T^a = \text{Tr} A \text{Tr} B - \frac{1}{N} \text{Tr} A B \end{aligned} \quad (\text{A.11})$$

Useful integrals for computing Feynman diagrams in momentum space and dimensional regularization ($d = 3 - 2\epsilon$) are, at one loop

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^2 (k-p)^2} = \frac{1}{8} \frac{1}{|p|} \equiv B_0(p) \quad (\text{A.12})$$

$$\int \frac{d^3 k}{(2\pi)^3} \frac{k_{\alpha\beta}}{k^2 (k-p)^2} = \frac{1}{2} p_{\alpha\beta} B_0(p) \quad (\text{A.13})$$

and at two loops

$$F(p) \equiv \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{1}{k^2 q^2 (p-k-q)^2} = \frac{\Gamma(\epsilon)}{64\pi^2} \sim \frac{1}{64\pi^2} \frac{1}{\epsilon} \quad (\text{A.14})$$

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